

Synchronised Similar Triangles for Three-Body Orbit with Zero Angular Momentum

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Abstract. Geometrical properties of three-body orbits with zero angular momentum are investigated.

If the moment of inertia is also constant along the orbit, the triangle whose vertexes are the positions of the bodies, and the triangle whose perimeters are the momenta of the bodies, are always similar (“synchronised similar triangles”). This similarity yields kinematic equalities between mutual distances and magnitude of momenta. Moreover, if the orbit is a solution to the equation of motion under homogeneous potential, the orbit has a new constant involving momenta.

For orbits with zero angular momentum and non-constant moment of inertia, we introduce scaled variables, positions divided by square root of the moment of inertia and momenta derived from the velocity of the scaled positions. Then the similarity and the kinematic equalities hold for the scaled variables. Using this similarity, we prove that any bounded three-body orbit with zero angular momentum under homogeneous potential whose degree is smaller than 2 has infinitely many collinear configurations (syzygies or eclipses) or collisions.

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1. Introduction and summary

Recently, the figure-eight solution to the planar equal-masses three-body problem was found by Moore, Chenciner & Montgomery and Simó [1, 2, 3, 4], and is paid attention

to. Numerical calculations shows that this solution is unique up to translation, rotation and scale transformation. Therefore, we write this solution “the figure-eight”.

The figure-eight solution has zero angular momentum. Fujiwara, Fukuda and Ozaki (FFO) [5] pointed out that this gives an important information of the orbit: three tangent lines at the three bodies meet at a point at each instant. Below we call this theorem the “three-tangents theorem”, and the crossing point the “centre of tangents” C_t . FFO used this theorem to find a figure-eight solution on the lemniscate curve [5].

Then natural questions arise. What will happen if three normal lines meet at a point? What conditions make three normal lines meet at a point? We show in this paper that in the planer three-body problem with general masses, if the moment of inertia is constant along the orbit, then three normal lines at the bodies meet at a point. We now call this theorem the “three normals theorem”, and the crossing point the “centre of normals” C_n .

The first half of this paper is devoted to clarify the nature of orbits with zero angular momentum and constant moment of inertia. These orbits have the following two geometrical properties. The first property is the following. The centre of tangents C_t and the centre of normals C_n are the end points of a diameter of the circumcircle for the triangle made by three bodies. Thus the midpoint of C_t and C_n is the circumcenter C_o . In figure 1, the figure-eight solution under the interaction potential $-1/r_{ij}^2$ is shown,

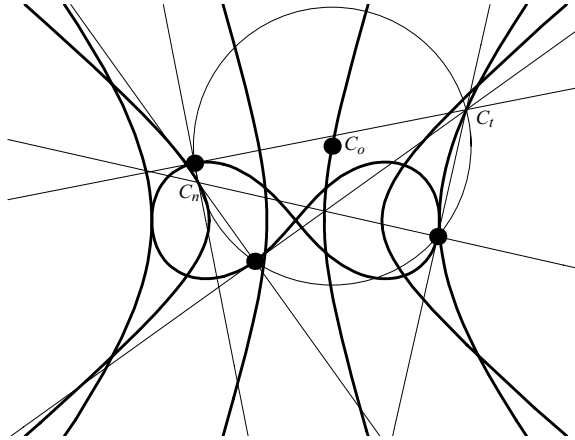


Figure 1. The eight-shaped curve is the figure-eight orbit under the potential $-1/r_{ij}^2$. Hyperbola-like curves from inner to outer are the orbits of C_o , C_n and C_t respectively. Tangent lines and normal lines are represented by thin lines.

where r_{ij} is the distance between bodies i and j . It is known that the figure-eight solution under this potential has zero angular momentum and a constant moment of inertia [6]. The positions of three bodies and the circumcenter C_o are represented by solid circles. The big circle is the circumcircle and we can find both the C_t and C_n on it, though the diameter between C_t and C_n are not shown.

The second property is a consequence of the first property. The triangle whose vertexes are the positions q_i , and the triangle whose perimeters are the momenta p_i , are

always inversely similar (similar in inverse orientation). In other words, the triangle whose perimeters are the mutual distances $r_{ij} = |q_i - q_j|$, and the triangle whose perimeters are the magnitude of the momenta $|p_k|$, are always inversely similar. See figure 2. We call these triangles the “synchronised similar triangles”, because they are always similar.

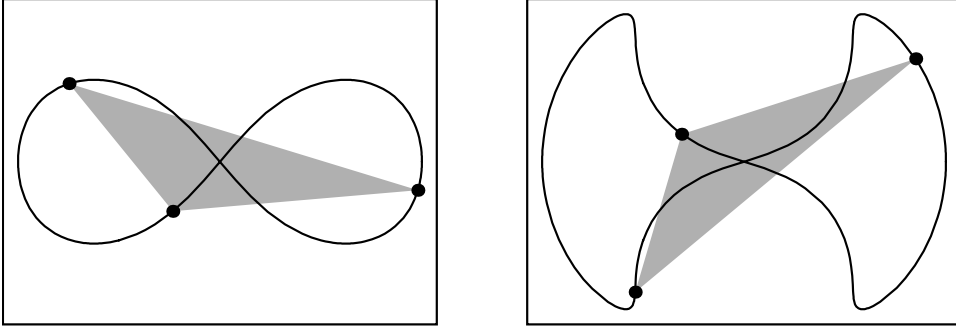


Figure 2. The figure-eight orbit under $-1/r_{ij}^2$ potential, with $m_i = 1$. Left: Orbit for $q'_i = q_i/\sqrt{I}$. Right: Orbit for $p'_k = (p_i - p_j)/\sqrt{3K}$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. These two triangles (grey areas) are always congruent with each other in inverse orientation.

We will show in the following that this similarity yields the following kinematic equalities

$$\frac{|p_1(t)|}{|q_2(t) - q_3(t)|} = \frac{|p_2(t)|}{|q_3(t) - q_1(t)|} = \frac{|p_3(t)|}{|q_1(t) - q_2(t)|} = \sqrt{\frac{m_1 m_2 m_3 K(t)}{MI}}, \quad (1)$$

$$\frac{q_i \wedge q_j}{I} + \frac{v_i \wedge v_j}{K} = 0 \quad (2)$$

where K , M and I represent twice of the kinetic energy, the total mass and the moment of inertia around the centre of mass, respectively. The symbol \wedge represents outer product. As shown in the equation (1), the mutual distances $r_{ij} = |q_i - q_j|$ and the magnitude of the momenta $|p_k|$ are strongly related.

Moreover, we will show that the solution orbit to the equation of motion under the homogeneous potential

$$V_\alpha = \begin{cases} \frac{1}{\alpha} \sum_{i < j} m_i m_j r_{ij}^\alpha & \text{for } \alpha \neq 0 \\ \sum_{i < j} m_i m_j \log r_{ij} & \text{for } \alpha = 0, \end{cases} \quad (3)$$

has a new constant along it:

$$\begin{cases} \sum_{(i,j,k)} m_i m_j |p_k|^\alpha = \text{constant} & \text{for } \alpha \neq 0 \\ \sum_{(i,j,k)} m_i m_j \log |p_k| = \text{constant} & \text{for } \alpha = 0, \end{cases} \quad (4)$$

where (i, j, k) runs for the cyclic permutations of $(1, 2, 3)$.

The latter half of this paper is devoted to clarify the nature of orbits with zero angular momentum and non-constant moment of inertia. For these orbits, consider the following scaled variables

$$\tilde{q}_i = \frac{q_i}{\sqrt{I}}, \quad (5)$$

$$\tilde{v}_i = \frac{d\tilde{q}_i}{dt}, \quad (6)$$

in order to make the scaled moment of inertia constant. Then the triangle whose vertexes are \tilde{q}_i , and the triangle whose perimeters are $m_k \tilde{v}_k$, are the “synchronised similar triangles”. The kinematic equalities (1) and (2) hold for these scaled variables, whereas the equation (4) does not. Using this similarity, we will get the following interesting equation

$$K q_i \wedge q_j + I v_i \wedge v_j = \frac{1}{2} \frac{dI}{dt} \frac{d}{dt} (q_i \wedge q_j). \quad (7)$$

This equation holds for any three-body orbit with zero angular momentum.

Then we will show that the oriented area

$$\Delta = \frac{1}{2} (q_2 - q_1) \wedge (q_3 - q_1), \quad (8)$$

satisfies the following equation under the potential V_α

$$I \frac{d}{dt} \left(\frac{1}{I} \frac{d\Delta}{dt} \right) = - \left(\frac{2K}{I} + \sum_{k\ell} (m_k + m_\ell) r_{k\ell}^{\alpha-2} \right) \Delta. \quad (9)$$

From this equation, we can easily prove that any bounded three-body orbit with zero angular momentum has infinitely many collinear configurations (syzygies or eclipses) or collisions, if $\alpha \leq 2$. This marvellous theorem was first formulated and proved by Montgomery [7], who derived an equation for Δ/I similar to the equation (9) with an elaborate calculation.

In section 2, we prove the four geometrical theorems, the “three tangents”, the “three normals”, the “circumcircle” and the “synchronised similar triangles”. In the same section, we also prove a purely algebraic theorem which is a generalisation of the theorem of “synchronised similar triangles”. In section 3, a new constant (4) is deduced along the orbit under a homogeneous potential. In section 4, we point out that there also exist “synchronised similar triangles” in the momentum space and the force space. The scaled variables (5) and (6) are introduced in section 5 and then another proof for Montgomery’s “infinitely many syzygies” [7] is given in section 6. Finally in section 7 we discuss some related problems.

2. Three tangents, three normals, circumcircle and synchronised similar triangles

We now show some geometric and kinematic properties of planar three-body orbits with zero angular momentum and constant moment of inertia.

Theorem 1 (Three Tangents) *If both the linear momentum and the angular momentum are zero, three tangent lines at the bodies meet at a point or three tangent lines are parallel.*

Proof: Assume two tangent lines at the bodies 1 and 2 meet at a point C_t . Since $\sum_i p_i = 0$ and $\sum_i q_i \wedge p_i = 0$, we have $\sum_i (q_i - C_t) \wedge p_i = 0$. By the assumption, we have $(q_1 - C_t) \wedge p_1 = (q_2 - C_t) \wedge p_2 = 0$. Thus we get $(q_3 - C_t) \wedge p_3 = 0$. That is, the tangent line at the body 3 also passes through the point C_t . Since $\sum_i p_i = 0$, it is obvious that if two tangent lines are parallel, the third line is also parallel to the other two lines.

Theorem 2 (Three Normals) *If the linear momentum is zero and the moment of inertia is constant, three normal lines at the bodies meet at a point or three normal lines are parallel.*

Proof: Similar argument holds for $\sum_i p_i = 0$ and $\sum_i q_i \cdot p_i = 0$.

Theorem 3 (Circumcircle) *If the linear momentum is zero, the angular momentum is zero and moment of inertia is constant, then the points C_t and C_n are the endpoints of a diameter of the circumcircle of the triangle made of q_1, q_2, q_3 .*

Proof: This is because the angles $C_t - q_i - C_n$ are 90 degrees for $i = 1, 2, 3$.

Theorem 4 (Synchronised Similar Triangles) *If the linear momentum is zero, the angular momentum is zero and moment of inertia is constant, then the triangle whose vertexes are q_i and the triangle whose perimeters are p_i , are always inversely similar.*

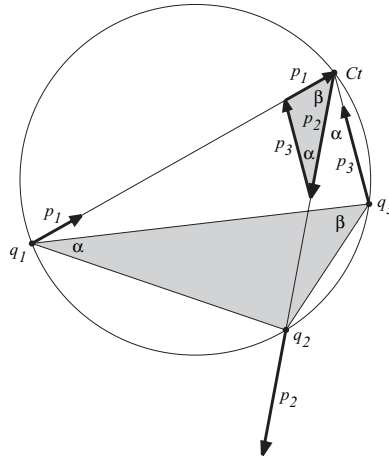


Figure 3. The triangle whose vertexes are q_i (large grey triangle) and the triangle whose perimeters are p_i (small grey triangle) are inversely similar.

Proof: Since $\sum_i p_i = 0$, vectors p_1, p_2, p_3 form a triangle. By the theorem 3, points q_1, q_2, q_3 and C_t are on the circumcircle. Then, the angles denoted by α in figure 3 are identical. The angles denoted by β are also identical.

We use the following notations

$$M = \sum_k m_k, \quad (10)$$

$$I = \sum_k m_k q_k^2 = M^{-1} \sum_{i < j} m_i m_j (q_i - q_j)^2, \quad (11)$$

$$K = \sum_k m_k v_k^2, \quad (12)$$

$$L = \sum_k q_k \wedge p_k, \quad (13)$$

and take the centre of mass to be the origin

$$\sum_k m_k q_k = 0. \quad (14)$$

Now let us prove the equation (1). Let $\kappa(t)$ be the ratio of magnification

$$\kappa(t) = \frac{|p_k(t)|}{|q_i(t) - q_j(t)|}. \quad (15)$$

Here $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$, and we always use this convention when indexes i, j, k appear in one equation. Then we get

$$\frac{\kappa^2}{m_1 m_2 m_3} = \frac{p_k^2/m_k}{m_i m_j (q_i - q_j)^2} = \frac{\sum_k p_k^2/m_k}{\sum_{i < j} m_i m_j (q_i - q_j)^2} = \frac{K}{MI}, \quad (16)$$

which leads to

$$\kappa = \sqrt{\frac{m_1 m_2 m_3 K}{MI}}, \quad (17)$$

i.e., the equation (1) and

$$\frac{m_i m_j (q_i - q_j)^2}{MI} = \frac{p_k^2/m_k}{K} = \frac{m_k v_k^2}{K}. \quad (18)$$

Now, consider the oriented area of the “synchronised similar triangles”. Since the two triangles are inversely similar, we get the equation (2) as follows

$$\begin{aligned} p_1 \wedge p_2 &= -\kappa^2 (q_2 - q_1) \wedge (q_3 - q_1) \\ &= -\frac{K}{MI} m_1 m_2 m_3 (q_1 \wedge q_2 + q_2 \wedge q_3 + q_3 \wedge q_1) \\ &= -\frac{K}{I} m_1 m_2 q_1 \wedge q_2, \end{aligned} \quad (19)$$

where we have used the relation of $m_1 m_2 q_1 \wedge q_2 = m_2 m_3 q_2 \wedge q_3 = m_3 m_1 q_3 \wedge q_1$, which follows from $\sum_i m_i q_i = 0$.

Note that the following identity holds for vectors η_i which satisfy $\sum_i m_i \eta_i = 0$,

$$m_i m_j (\eta_i - \eta_j)^2 + M m_k \eta_k^2 = (m_i + m_j) \sum_\ell m_\ell \eta_\ell^2. \quad (20)$$

Using this identity, the equation (18) can be written as

$$\frac{m_i m_j (v_i - v_j)^2}{MK} = \frac{m_k v_k^2}{I}. \quad (21)$$

Then, the equations (18), (21) and (20) yield the following interesting equation

$$\frac{m_k q_k^2}{I} + \frac{m_k v_k^2}{K} = \frac{m_i m_j (q_i - q_j)^2}{MI} + \frac{m_i m_j (v_i - v_j)^2}{MK} = \frac{m_i + m_j}{M}. \quad (22)$$

The equation (21) shows that

$$\frac{m_k |q_k|}{|v_i - v_j|} = \sqrt{\frac{m_1 m_2 m_3 I}{MK}}. \quad (23)$$

Therefore, with the equation (2), we conclude that the triangle whose vertexes are v_i and the triangle whose perimeters are $m_k q_k$, are always inversely similar. Thus, the role of q_i and v_i are completely equivalent. Indeed, the following purely algebraic theorem holds.

Theorem 5 Consider two triplets of two dimensional vectors $\{\xi_i\}$, $\{\bar{\xi}_i\}$ and a triplet of scalars $\{\mu_i\}$, $i = 1, 2, 3$, which satisfy

$$\sum_i \mu_i \xi_i = 0, \quad \sum_i \mu_i \bar{\xi}_i = 0, \quad \sum_i \mu_i \xi_i \wedge \bar{\xi}_i = 0, \quad \sum_i \mu_i \xi_i \cdot \bar{\xi}_i = 0. \quad (24)$$

Let I be the “moment function” defined by

$$I(\eta) = \sum_i \mu_i \eta_i^2. \quad (25)$$

Then, we have the following three equivalent equations

$$\frac{\mu_k \xi_k^2}{I(\xi)} = \frac{\mu_i \mu_j (\bar{\xi}_i - \bar{\xi}_j)^2}{(\mu_1 + \mu_2 + \mu_3) I(\bar{\xi})}, \quad (26)$$

$$\frac{\mu_i \mu_j (\xi_i - \xi_j)^2}{(\mu_1 + \mu_2 + \mu_3) I(\xi)} = \frac{\mu_k \bar{\xi}_k^2}{I(\bar{\xi})}, \quad (27)$$

$$\frac{\mu_k \xi_k^2}{I(\xi)} + \frac{\mu_k \bar{\xi}_k^2}{I(\bar{\xi})} = \frac{\mu_i \mu_j (\xi_i - \xi_j)^2}{(\mu_1 + \mu_2 + \mu_3) I(\xi)} + \frac{\mu_i \mu_j (\bar{\xi}_i - \bar{\xi}_j)^2}{(\mu_1 + \mu_2 + \mu_3) I(\bar{\xi})} = \frac{\mu_i + \mu_j}{\mu_1 + \mu_2 + \mu_3} \quad (28)$$

and

$$\frac{\xi_i \wedge \xi_j}{I(\xi)} + \frac{\bar{\xi}_i \wedge \bar{\xi}_j}{I(\bar{\xi})} = 0. \quad (29)$$

Remark Therefore, triangle whose vertexes are ξ_i , and triangle whose perimeters are $\mu_i \bar{\xi}_i$, are inversely similar. Equivalently, triangle whose vertexes are $\bar{\xi}_i$, and triangle whose perimeters are $\mu_i \xi_i$, are also inversely similar.

Proof: Let us denote the Jacobi coordinates in ξ space and $\bar{\xi}$ space respectively by $\{a, b\}$ and $\{\bar{a}, \bar{b}\}$, where

$$a = \rho \frac{\mu_1 \xi_1 + \mu_2 \xi_2}{\mu_1 + \mu_2} = -\rho \frac{\mu_3 \xi_3}{\mu_1 + \mu_2}, \quad (30)$$

$$b = \sigma (\xi_1 - \xi_2) \quad (31)$$

with similar expressions for the “bar” variables, and

$$\rho = \sqrt{\frac{(\mu_1 + \mu_2)(\mu_1 + \mu_2 + \mu_3)}{\mu_3}}, \quad \sigma = \sqrt{\frac{\mu_1 \mu_2}{\mu_1 + \mu_2}}. \quad (32)$$

The inverse relations are

$$\xi_1 = \frac{a}{\rho} + \frac{\mu_2}{\mu_1 + \mu_2} \frac{b}{\sigma}, \quad (33)$$

$$\xi_2 = \frac{a}{\rho} - \frac{\mu_1}{\mu_1 + \mu_2} \frac{b}{\sigma}, \quad (34)$$

$$\xi_3 = -\frac{\mu_1 + \mu_2}{\mu_3} \frac{a}{\rho} \quad (35)$$

and similar expressions for the “bar” variables. Then, the equations $\sum_i \mu_i \xi_i = 0$, $\sum_i \mu_i \bar{\xi}_i = 0$ are automatically satisfied and the equations $\sum_i \mu_i \xi_i \wedge \bar{\xi}_i = 0$, $\sum_i \mu_i \xi_i \cdot \bar{\xi}_i = 0$ yield

$$a \wedge \bar{a} + b \wedge \bar{b} = 0, \quad (36)$$

$$a \cdot \bar{a} + b \cdot \bar{b} = 0. \quad (37)$$

Let us use the polar coordinate for the vectors a, b, \bar{a}, \bar{b} :

$$a = (|a|, \alpha), b = (|b|, \beta) \quad (38)$$

and similar notations for the “bar” variables. The equations (36) and (37) then yield

$$|a| |\bar{a}| \sin(\bar{\alpha} - \alpha) + |b| |\bar{b}| \sin(\bar{\beta} - \beta) = 0, \quad (39)$$

$$|a| |\bar{a}| \cos(\bar{\alpha} - \alpha) + |b| |\bar{b}| \cos(\bar{\beta} - \beta) = 0, \quad (40)$$

which give

$$a^2 \bar{a}^2 = b^2 \bar{b}^2, \quad (41)$$

$$\bar{\beta} - \beta = \bar{\alpha} - \alpha + \pi. \quad (42)$$

From the equation (41) we get

$$\frac{\bar{b}^2}{a^2} = \frac{\bar{a}^2}{b^2} = \frac{\bar{a}^2 + \bar{b}^2}{a^2 + b^2}, \quad (43)$$

where the second equality is a consequence of the first equality. Then, we have the following three equivalent equations

$$\frac{a^2}{a^2 + b^2} = \frac{\bar{b}^2}{\bar{a}^2 + \bar{b}^2}, \quad (44)$$

$$\frac{b^2}{a^2 + b^2} = \frac{\bar{a}^2}{\bar{a}^2 + \bar{b}^2}, \quad (45)$$

$$\frac{a^2}{a^2 + b^2} + \frac{\bar{a}^2}{\bar{a}^2 + \bar{b}^2} = \frac{b^2}{a^2 + b^2} + \frac{\bar{b}^2}{\bar{a}^2 + \bar{b}^2} = 1. \quad (46)$$

Note that $a^2 + b^2 = \sum_i \mu_i \xi_i^2 = I(\xi)$ is the “moment function” defined in the theorem 5. Rewriting the equations (44)–(46) by ξ and $\bar{\xi}$ variables, we get the equations (26)–(28) in the theorem 5.

From the equation (42) we get

$$\bar{\beta} - \bar{\alpha} = \beta - \alpha + \pi. \quad (47)$$

Moreover, the product of equations (44) and (45) yield

$$\frac{|a| |b|}{a^2 + b^2} = \frac{|\bar{a}| |\bar{b}|}{\bar{a}^2 + \bar{b}^2}. \quad (48)$$

Therefore, the following equation is obvious

$$\frac{a \wedge b}{a^2 + b^2} + \frac{\bar{a} \wedge \bar{b}}{\bar{a}^2 + \bar{b}^2} = \frac{|a| |b| \sin(\beta - \alpha)}{a^2 + b^2} + \frac{|\bar{a}| |\bar{b}| \sin(\bar{\beta} - \bar{\alpha})}{\bar{a}^2 + \bar{b}^2} = 0. \quad (49)$$

Rewriting this equation by ξ and $\bar{\xi}$, we finally get the equation (29) in the theorem 5.

3. Constant along the orbit under homogeneous potentials

Let us consider the orbit with zero angular momentum and constant moment of inertia under the potential V_α defined by the equation (3).

Since $I = \text{constant}$, the Jacobi-Lagrange identity yields

$$0 = \frac{d^2 I}{dt^2} = \begin{cases} 2(K - \alpha V_\alpha) = 2 \left(K - \sum_{i < j} m_i m_j r_{ij}^\alpha \right) & \text{for } \alpha \neq 0 \\ 2 \left(K - \sum_{i < j} m_i m_j \right) & \text{for } \alpha = 0. \end{cases} \quad (50)$$

Therefore for $\alpha \neq -2$, the constant moment of inertia along the orbit yields

$$K = \begin{cases} \sum_{i < j} m_i m_j r_{ij}^\alpha = \frac{\alpha}{2 + \alpha} E & \text{for } \alpha \neq 0, -2 \\ \sum_{i < j} m_i m_j = 2 \left(E - \sum_{ij} m_i m_j \log r_{ij} \right) & \text{for } \alpha = 0, \end{cases} \quad (51)$$

and both the kinetic energy $K/2$ and the potential energy V_α are constant. As a consequence of this equation, r_{ij} cannot be zero if $\alpha \leq 0$ and $\alpha \neq -2$. That is, there is no collision along the orbit. For $\alpha = -2$, on the other hand, K and V_{-2} can vary along the orbit keeping the energy balance

$$K = -2V_{-2} = \sum_{i < j} \frac{m_i m_j}{r_{ij}^2} \text{ for } \alpha = -2, \quad (52)$$

with zero total energy.

Moreover, if the angular momentum is zero, then by the use of the equation (1), r_{ij}^2 can be expressed by the momentum p_k^2 ,

$$r_{ij}^2 = \frac{MI}{m_1 m_2 m_3 K} p_k^2. \quad (53)$$

Then the equations (51) or (52) yields

$$\begin{cases} \sum_{(ijk)} m_i m_j |p_k|^\alpha = K \left(\frac{m_1 m_2 m_3 K}{MI} \right)^{\frac{\alpha}{2}} & \text{for } \alpha \neq 0 \\ \sum_{(ijk)} m_i m_j \log |p_k| = E + \frac{K}{2} \log \frac{m_1 m_2 m_3 K}{MI} & \text{for } \alpha = 0. \end{cases} \quad (54)$$

The right-hand sides of the above equations are constant for all α . Note that for $\alpha = -2$,

$$\text{r.h.s.} = \frac{MI}{m_1 m_2 m_3}, \quad (55)$$

is also constant along the orbit. Thus we get the equation (4). As a consequence of this equation, p_k cannot be zero if $\alpha \leq 0$, meaning that the bodies cannot stop.

For $\alpha \neq -2$, the orbit of $L = 0$ and $I = \text{constant}$ is strictly constrained by the condition for K and V_α to be constant and the condition (54). For $\alpha = -2$, on the other hand, the condition (52) is relatively loose and actually the figure-eight orbit shown in figure 1 is that with $L = 0$ and $I = \text{constant}$.

4. Similarity in the momentum space and the force space

The similarity for the momentum space and the force space also holds. This is interesting, but this similarity does not produce any new information.

Differentiate the equation (54) with respect to t , we get

$$m_1 m_2 |p_3|^{\alpha-2} p_3 \cdot f_3 + m_2 m_3 |p_1|^{\alpha-2} p_1 \cdot f_1 + m_3 m_1 |p_2|^{\alpha-2} p_2 \cdot f_2 = 0, \quad (56)$$

where

$$f_k = \frac{dp_k}{dt} \quad (57)$$

represents the force acting on the body k . Substituting the equation (1) into the above equation, we get

$$m_1 m_2 r_{12}^{\alpha-2} p_3 \cdot f_3 + m_2 m_3 r_{23}^{\alpha-2} p_1 \cdot f_1 + m_3 m_1 r_{31}^{\alpha-2} p_2 \cdot f_2 = 0. \quad (58)$$

On the other hand, we have the following equality

$$m_1 m_2 r_{12}^{\alpha-2} p_3 \wedge f_3 + m_2 m_3 r_{23}^{\alpha-2} p_1 \wedge f_1 + m_3 m_1 r_{31}^{\alpha-2} p_2 \wedge f_2 = 0, \quad (59)$$

which is proved by substituting the equations of motion into the forces f_k and showing that l.h.s. becomes

$$m_1 m_2 m_3 \left(\sum_k q_k \wedge p_k \right) (m_1 r_{31}^{\alpha-2} r_{12}^{\alpha-2} + m_2 r_{12}^{\alpha-2} r_{23}^{\alpha-2} + m_3 r_{23}^{\alpha-2} r_{31}^{\alpha-2}) = 0. \quad (60)$$

Let

$$\mu_1^{-1} = m_2 m_3 r_{23}^{\alpha-2}, \quad \mu_2^{-1} = m_3 m_1 r_{31}^{\alpha-2}, \quad \mu_3^{-1} = m_1 m_2 r_{12}^{\alpha-2} \quad (61)$$

and

$$\xi_i = \frac{p_i}{\mu_i}, \quad \bar{\xi}_i = \frac{f_i}{\mu_i}. \quad (62)$$

Then, the equations $\sum p_i = 0$, $\sum f_i = 0$, (58) and (59) are rewritten as

$$\sum_i \mu_i \xi_i = 0, \quad \sum_i \mu_i \bar{\xi}_i = 0, \quad \sum_i \mu_i \xi_i \cdot \bar{\xi}_i = 0, \quad \sum_i \mu_i \xi_i \wedge \bar{\xi}_i = 0. \quad (63)$$

By the theorem 5, the triangle whose vertexes are $\bar{\xi}_k = f_k / \mu_k = m_1 m_2 m_3 r_{ij}^{\alpha-2} d^2 q_k / dt^2$, and the triangle whose perimeters are $\mu_i \xi_i = p_i$, are the “synchronised similar triangles”. They are always inversely similar. However, this similarity gives no new information, because they are equivalent to the similarity in $q-v$ variables.

5. Synchronised similar triangles for $L = 0$ orbit

In this section, we consider general three-body orbits with $L = 0$, but not with the assumption of $I = \text{constant}$. Even in this case, we can find the “synchronised similar triangles”. Consider the scaled position and the velocity of the scaled position defined by the equations (5) and (6). We can easily verify the following equalities for the scaled variables,

$$\sum_i m_i \tilde{q}_i = 0, \quad \sum_i m_i \tilde{v}_i = 0, \quad \sum_i m_i \tilde{q}_i \wedge \tilde{v}_i = 0, \quad \sum_i m_i \tilde{q}_i \cdot \tilde{v}_i = 0. \quad (64)$$

By the theorem 5, triangle whose vertexes are \tilde{q}_i , and triangle whose perimeters are $m_i \tilde{v}_i$, are the “synchronised similar triangles”. Therefore, all the equalities in the section 2 hold for the variables \tilde{q}_i and \tilde{v}_i . Rewriting the equality for \tilde{q}_i and \tilde{v}_i by the original variables q_i and v_i , we have useful equalities for $L = 0$ and $I \neq \text{constant}$ orbits. For example, the equality (2) for the scaled variables

$$\tilde{q}_i \wedge \tilde{q}_j + \frac{\tilde{v}_i \wedge \tilde{v}_j}{\sum_k m_k \tilde{v}_k^2} = 0 \quad (65)$$

yields the equality (7), since

$$\tilde{q}_i \wedge \tilde{q}_j = \frac{q_i \wedge q_j}{I}, \quad (66)$$

$$\tilde{v}_i \wedge \tilde{v}_j = \frac{1}{I} \left(v_i \wedge v_j + q_i \wedge q_j \frac{1}{4I^2} \left(\frac{dI}{dt} \right)^2 - \frac{1}{2I} \frac{dI}{dt} \frac{d}{dt} (q_i \wedge q_j) \right), \quad (67)$$

$$\sum_k m_k \tilde{v}_k^2 = \frac{K}{I} - \frac{1}{4I^2} \left(\frac{dI}{dt} \right)^2. \quad (68)$$

6. Infinitely many syzygies or collisions for $L = 0$ orbit

In this section, we consider $L = 0$ orbit under the potential energy V_α . We do not assume $I = \text{constant}$. We derive an equation of motion for the oriented area defined by the positions q_i of three bodies, and prove that any three-body orbits with $L = 0$ under the potential V_α with $\alpha \leq 2$ have infinitely many syzygies or collisions.

Let $\Delta = 2^{-1}(q_2 - q_1) \wedge (q_3 - q_1)$ be the oriented area and

$$\Lambda_{ij} = q_i \wedge q_j \quad (69)$$

be twice of the oriented area of the triangle defined by q_i , q_j and the origin. Note that

$$\Delta = \frac{1}{2} (\Lambda_{12} + \Lambda_{23} + \Lambda_{31}). \quad (70)$$

The second derivative of Λ_{ij} with respect to time t is

$$\frac{d^2 \Lambda_{ij}}{dt^2} = 2 v_i \wedge v_j + \frac{d^2 q_i}{dt^2} \wedge q_j + q_i \wedge \frac{d^2 q_j}{dt^2}. \quad (71)$$

By virtue of (7), we have

$$v_i \wedge v_j = -\frac{K}{I} \Lambda_{ij} + \frac{1}{2I} \frac{dI}{dt} \frac{d\Lambda_{ij}}{dt}. \quad (72)$$

Using the equation of motion for q_i , we get

$$\frac{d^2 q_i}{dt^2} \wedge q_j + q_i \wedge \frac{d^2 q_j}{dt^2} = -\Lambda_{ij} \sum_{k\ell} (m_k + m_\ell) r_{k\ell}^{\alpha-2}. \quad (73)$$

Substituting the equations (72) and (73) into the equation (71), we get the equation for Λ_{ij}

$$\frac{d^2 \Lambda_{ij}}{dt^2} = - \left(\frac{2K}{I} + \sum_{k\ell} (m_k + m_\ell) r_{k\ell}^{\alpha-2} \right) \Lambda_{ij} + \frac{1}{I} \frac{dI}{dt} \frac{d\Lambda_{ij}}{dt}, \quad (74)$$

and the equation for Δ of the same form,

$$\frac{d^2 \Delta}{dt^2} = - \left(\frac{2K}{I} + \sum_{k\ell} (m_k + m_\ell) r_{k\ell}^{\alpha-2} \right) \Delta + \frac{1}{I} \frac{dI}{dt} \frac{d\Delta}{dt}, \quad (75)$$

which is equivalent to the equation (9).

Now, let us prove that the function $\Delta(t)$ or equivalently

$$S(t) = \frac{\Delta(t)}{\sqrt{I(t)}} \quad (76)$$

has infinitely many zeros for $\alpha \leq 2$. Note that the zeros of $\Delta(t)$ and $S(t)$ are one to one. There are three cases when they take zero: syzygy without collision, two-body collision and triple collision $I \rightarrow 0$. Thus, infinitely many zeros of $S(t)$ correspond to infinitely many syzygies or collisions of the orbit. Montgomery proved this property making use of the equation of motion for Δ/I . We give here another proof from a different point of view. Our proof consists of three steps.

The first step: Eliminate the first derivative term $d\Delta/dt$ in the equation (75). To do this, we consider the equation for $S(t)$ instead of $\Delta(t)$. Then, the equation (75) is equivalent to

$$\begin{aligned} \frac{d^2 S}{dt^2} &= - \left\{ \sum_{i < j} (m_i + m_j) r_{ij}^{\alpha-2} + \frac{2K}{I} + \frac{1}{2I} \frac{d^2 I}{dt^2} - \frac{3}{4I^2} \left(\frac{dI}{dt} \right)^2 \right\} S \\ &= - \left\{ \frac{M}{I} \sum_{(ijk)} m_k q_k^2 r_{ij}^{\alpha-2} + \frac{3K}{I} - \frac{3}{4I^2} \left(\frac{dI}{dt} \right)^2 \right\} S, \end{aligned} \quad (77)$$

where we have used two identities $d^2 I / dt^2 = 2(K - \alpha V_\alpha)$ and (20) for $m_i + m_j$.

The second step: Write the equation (77) as

$$\frac{d^2 S}{dt^2} = -\omega^2 S, \quad (78)$$

$$\omega^2 = \frac{M}{I} \sum_{(ijk)} \frac{m_k q_k^2}{r_{ij}^{2-\alpha}} + \frac{3K}{I} - \frac{3}{4I^2} \left(\frac{dI}{dt} \right)^2, \quad (79)$$

and show that ω^2 is bounded from below by a positive constant $\omega^2 \geq \omega_0^2 > 0$ for $\alpha \leq 2$.

The following inequalities hold

$$\left(\frac{dI}{dt} \right)^2 = \left(2 \sum_i m_i q_i v_i \right)^2 \leq 4 \left(\sum_i m_i q_i^2 \right) \left(\sum_i m_i v_i^2 \right) = 4IK, \quad (80)$$

$$\sum_{(ijk)} \frac{m_k q_k^2}{r_{ij}^{2-\alpha}} \geq \sum_{(ijk)} m_k q_k^2 \left(\frac{m_{\min}^2}{MI} \right)^{(2-\alpha)/2} = I \left(\frac{m_{\min}^2}{MI} \right)^{(2-\alpha)/2}, \quad (81)$$

where we have used an inequality that $MI = \sum_{i < j} m_i m_j r_{ij}^2 \geq m_i m_j r_{ij}^2 \geq m_{\min}^2 r_{ij}^2$ and $2 \geq \alpha$. The symbol m_{\min} represents the minimum value of $\{m_k\}$. Then, we get the following inequality

$$\omega^2 \geq M \left(\frac{m_{\min}^2}{MI} \right)^{(2-\alpha)/2} \geq M \left(\frac{m_{\min}^2}{MI_{\max}} \right)^{(2-\alpha)/2} = \omega_0^2 > 0 \quad (82)$$

where we have used the fact that the orbit is bounded $I \leq I_{\max}$.

The last step: Since the restoring force for $S(t)$ is always stronger than that of a harmonic oscillator with $\omega = \omega_0$, it is natural to expect that $S(t)$ has infinitely many zeros and intervals of zeros are shorter than $T_0 = \pi/\omega_0$. To prove this we show that for any initial conditions $S(0)$ and $dS/dt(0)$, the function $S(t)$ vanishes before $t = T_0$. Let us consider a harmonic oscillator $A(t)$ which has the period $2T_0 = 2\pi/\omega_0$ and satisfies the same initial conditions as $S(t)$,

$$\frac{d^2 A(t)}{dt^2} = -\omega_0^2 A(t), \quad (83)$$

$$A(0) = S(0), \quad \frac{dA}{dt}(0) = \frac{dS}{dt}(0). \quad (84)$$

Define a function $Z(t)$ by

$$Z(t) = S(t) \frac{dA(t)}{dt} - \frac{dS(t)}{dt} A(t). \quad (85)$$

Then, the first derivative with respect to t yields

$$\frac{dZ(t)}{dt} = S(t) \frac{d^2 A(t)}{dt^2} - \frac{d^2 S(t)}{dt^2} A(t) = (\omega^2 - \omega_0^2) S(t) A(t). \quad (86)$$

Without loss of generality, we can take $A(0) = S(0) > 0$. Let $t_0 > 0$ be the first time when $A(t)$ vanishes. Then we have $t_0 < T_0$ and $dA/dt(t_0) < 0$. To derive a contradiction, suppose that $S(t) > 0$ for $0 \leq t \leq t_0$. Then, $(\omega^2 - \omega_0^2) S(t) A(t) \geq 0$ for $0 \leq t \leq t_0$. Since $Z(0) = 0$ and $A(t_0) = 0$, integration of the equation (86) yields

$$Z(t_0) = S(t_0) \frac{dA}{dt}(t_0) = \int_0^{t_0} (\omega^2 - \omega_0^2) S(s) A(s) ds \geq 0. \quad (87)$$

Since $dA(t)/dt(t_0) < 0$, we get $S(t_0) \leq 0$, which is a contradiction. Therefore, $S(t)$ must have zero before $t = t_0 < T_0$.

7. Final remarks

This work may give rich information for some related problems. The first one is to investigate the nature of the figure-eight solution under $-1/r_{ij}^2$ potential which has zero angular momentum and constant moment of inertia. We hope that the equalities (1), (2) and (4) would be useful for further understanding of the figure-eight solution under this potential.

The second problem is to find a conceptual proof of Chenciner's problem 13 in his lecture at Taiyuan [6]. Consider the figure-eight solution under the homogeneous or logarithmic potential energy V_α . Chenciner's problem is the following: *Show that the moment of inertia of the "Eight" stays constant only when $\alpha = -2$.* In the equal-masses three-body problem, FFO [8] showed that a motion satisfying the following three conditions exists under the potentials V_α if and only if $\alpha = -2, 2, 4$. The conditions are (i) $L = 0$, (ii) $I = \text{constant}$ and (iii) one body passes through the centre of mass. Then FFO explicitly proved that the orbits for $\alpha = 2, 4$ are not the figure-eight. This solves Chenciner's problem. But the method of FFO was a kind of "brute-force" one, in which they calculated the derivative of the moment of inertia with respect to time to the eighth order explicitly. A more "conceptual" proof would be appreciated. The present work shows what will happen if $L = 0$ and $I = \text{constant}$ orbit exist for V_α with $\alpha \neq -2$. As discussed in the section 3, these orbits are strictly constrained.

The third problem is to understand the nature of three-body orbits with zero angular momentum under various potential energy V_α . We expect that the information for the scaled variables will be useful for this problem.

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